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Transient conduction through a two-layer medium

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NOMENCLATURE

a	thickness of region 1
c	specific heat
erfc (z)	$(2/\pi^{1/2})\int_z^\infty e^{-\gamma^2}\,d\gamma$
$I(z)$	$(1/\pi^{1/2}z)\exp(-z^2)-\text{erfc}(z)$
k	$(\rho c/\lambda)^{1/2}$
p	Laplace transform operator
Q	heat flux
t	time
T	temperature
x	space coordinate.

Greek symbols

β	effusivity ratio, $(k_2\lambda_2/k_1\lambda_1)=(\rho_2c_2\lambda_2/\rho_1c_1\lambda_1)^{1/2}$
θ_1	λ_1T_1/Qa
θ_2	λ_1T_2/Qa
λ	thermal conductivity
ξ	$(k_2/k_1)(x/a-1)$
ρ	density
τ	$2t^{1/2}/k_1a$
χ	x/a.

1. INTRODUCTION

THE PROBLEM of transient conduction through composite media is of great practical interest.

The use of composite media is necessary when the thermal and mechanical properties of a single medium are not appropriate to fulfil both thermal and mechanical requirements. Nature is such that very good insulators do not have high mechanical strength and that materials of high mechanical strength are not very good insulators.

The range of heat fluxes encountered in applications is very wide, going from 0.1 W cm^{-2} for the natural solar flux on the earth to 10^8 W cm^{-2} for high energy lasers. We became interested in studying the problem of transient heat conduction while carrying out research on a fluidic igniter [1, 2]. The device uses the heat flux (about 10^2 W cm^{-2}) produced near the endwall of a Hartmann–Sprenger tube to ignite a pyrotechnic composition. Hence, it is important to know how fast a given layer of the composition, which is protected by a thin metal sheet is heated to the ignition temperature.

In this note, the problem of transient conduction through a medium composed of a semi-infinite layer protected by a layer of finite thickness exposed to a constant heat flux is studied. The solution of this transient heat conduction may be written in the form of Laplace transforms. However, the inverse transform brings some difficulties and earlier papers propose various approximate solutions for particular cases. Griffith and Horton [3] study the rise of temperature of insulated wall surfaces in a domestic room. They use a 'short time' approximation by a series expansion to invert the Laplace transform. Therefore, their solution has a limited range of application. Moreover, they did not use a non-dimensional approach, a form which allows information to be displayed more efficiently and in a more general fashion. Carslaw and Jaeger [4] indicate a general procedure to invert the Laplace transform but do not study the case when a constant flux is applied to the external face of the medium.

In calculating the response time of a heat film gauge, Maulard [5] finds an approximation of the inverse transform by observing that the thermal effusivity of the thin film is much larger than that of its support.

The precise range of validity of the various approximate solutions mentioned above is difficult to assess without the possibility of comparing them with an exact solution. For some applications, as the one we were interested in, previously published approximations were found to be inadequate.

The purpose of the present note is two fold. First, to give an exact solution to the problem in dimensionless form using contour integration in the complex plane. This method does not appear to have been previously used for the case of a constant applied heat flux. Second, to express the solution in power series to cover the whole range of the effusivity ratio of the two media.

The numerical values of the solutions given by the two methods are compared and the validity range of the various power series is given.

2. BASIC RESULTS

Our aim is to discuss a one-dimensional unsteady problem of heat conduction, and we denote by x, t the space and time coordinates, respectively. The regions $0 \leqslant x \leqslant a$ and $a \leqslant x \leqslant \infty$ are occupied by two homogeneous conducting materials whose properties are referred to by the subscripts 1 and 2, respectively. For $t < 0$ the ambient temperature is constant and the materials are in thermal equilibrium. At $t = 0$ a prescribed flux of heat, say Q, is applied at $x = 0$, and the aim is to calculate the temperature in regions 1 and 2 for $t > 0$.

The boundary conditions are that the heat flux must equal Q at $x = 0$ and that both the temperature T and the heat flux are continuous at $x = a$. Elsewhere T must satisfy the usual equation for heat conduction, namely

$$\frac{\lambda}{\rho c}\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial t} \tag{1}$$

where λ, ρ, and c are the thermal conductivity, density, and specific heat of the relevant material, respectively.*

* *Note on the use of the Laplace transform*: the transform $\bar{T}(p)$ of a function (t) and the inverse transform are defined respectively as

$$\bar{T}(p)=\int_0^\infty e^{-pt}\,T(t)\,dt$$

$$T(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\,\bar{T}(p)\,dp$$

where the contour of the integral is chosen so that all singularities of the integral are to the left of the contour. In the body of the text the bar over the transform is omitted and no distinction of notation is made between a function and its transform. This does not cause confusion where the context makes clear the distinction. The advantages are economy of notation, simplicity of presentation and ease of identification. Note, however, that the transform of a constant Q is Q/p.

A solution of equation (1) is sought for both T_1 and T_2 which satisfies the following boundary conditions for the *Laplace transform* of the temperature

$$\frac{Q}{p} = -\lambda_1 \frac{\partial T}{\partial x} \quad \text{at} \quad x = 0$$
$$T_1 = T_2, \lambda_1 \frac{\partial T_1}{\partial x} = \lambda_2 \frac{\partial T_2}{\partial x} \quad \text{at} \quad x = a. \tag{2}$$

Note the previous warning that the transform of a constant Q is Q/p.

Equation (1) has solutions of the form exp $(\pm kp^{1/2})$, where

$$k = \left(\frac{\rho c}{\lambda}\right)^{1/2} \tag{3}$$

and the general solution for T_1 is a linear combination of such solutions. For $T_2 \to 0$ as $x \to \infty$ (the temperature is measured in excess of the ambient temperature). The constants of integration are to be adjusted so that the boundary conditions (2) are satisfied, and it is a straightforward matter to verify that the following expressions satisfy both equations (1) and (2)

$$T_1 = \frac{Q}{p^{3/2}k_1\lambda_1} \frac{(1+\beta)\,e^{p^{1/2}k_1(a-x)} + (1-\beta)\,e^{-p^{1/2}k_1(a-x)}}{(1+\beta)\,e^{p^{1/2}k_1a} - (1-\beta)\,e^{-p^{1/2}k_1a}} \tag{4}$$

$$T_2 = \frac{Q}{p^{3/2}k_1\lambda_1} \frac{2\,e^{-p^{1/2}k_2(x-a)}}{(1+\beta)\,e^{p^{1/2}k_1a} - (1-\beta)\,e^{-p^{1/2}k_1a}} \tag{5}$$

where

$$\beta = \frac{k_2\lambda_2}{k_1\lambda_1} = \left(\frac{\rho_2 c_2 \lambda_2}{\rho_1 c_1 \lambda_1}\right)^{1/2}.$$

2.1. *Asymptotic behaviour*

Equations (4) and (5) cannot be interpreted in terms of elementary functions of t, but there are simple asymptotic solutions. Expansions for small values of t can be deduced from the behaviour of expressions (4) and (5) for $|p| \gg 1$. Thus we have

$$T_1 = \frac{Q}{p^{3/2}k_1\lambda_1}\left[e^{-p^{1/2}k_1x} + \sum_{n=1}^{\infty}\left(\frac{1-\beta}{1+\beta}\right)^n \times \left(\exp\{-p^{1/2}k_1(2na+x)\} + \exp\{-p^{1/2}k_1(2na-x)\}\right)\right]. \tag{7}$$

When this is interpreted† and expressed in terms of the non-dimensional variables

$$\chi = \frac{x}{a}, \quad \tau = \frac{2t^{1/2}}{k_1a} \tag{8}$$

the result is

$$\Theta_1 = \frac{\lambda_1 T_1}{Qa} = \chi I\left(\frac{\chi}{\tau}\right) + \sum_{n=1}^{\infty}\left(\frac{1-\beta}{1+\beta}\right)^n \times \left\{(2n+\chi) I\left(\frac{2n+\chi}{\tau}\right) + (2n-\chi) I\left(\frac{2n-\chi}{\tau}\right)\right\}. \tag{9}$$

In a similar fashion, we have from equation (5)

$$T_2 = \frac{2Q}{p^{3/2}k_1\lambda_1(1+\beta)} \sum_{n=0}^{\infty}\left(\frac{1-\beta}{1+\beta}\right)^n \times \exp\{-p^{1/2}k_1(x-a) - (2n+1)p^{1/2}k_1a\} \tag{10}$$

$$\Theta_2 = \frac{\lambda_1 T_2}{Qa} = \frac{2}{1+\beta}\sum_{n=0}^{\infty}\left(\frac{1-\beta}{1+\beta}\right)^n (2n+1+\xi) I\left(\frac{2n+1+\xi}{\tau}\right) \tag{11}$$

where

$$\xi = \frac{k_2}{k_1}(x-a). \tag{12}$$

The above expressions are exact, but they will not be of practical use when $\tau \gg 1$. For large τ the temperature is determined by the behaviour of expressions (4) and (5) in the neighbourhood of the origin in the complex plane. For small $|p|$ equation (4) gives

$$T_1 = \frac{Q}{p^{3/2}k_1\lambda_1} \frac{1+\beta p^{1/2}k_1(a-x) + \frac{1}{2}pk_1^2(a-x)^2}{\beta + p^{1/2}k_1a + \frac{1}{2}\beta p_1 k_1^2 a^2}$$
$$= \frac{Q}{p^{3/2}k_1\lambda_1}\left[\frac{1+\beta p^{1/2}k_1(a-x) + \frac{1}{2}pk_1^2(a-x)^2}{\beta + p^{1/2}k_2 a} - \frac{\beta p k_1^2 a^2}{2(\beta + p^{1/2}k_1a)^2}\right] \tag{13}$$

approximately. The interpretation of this expression then gives, for $\tau \gg 1$

$$\Theta_1 = \frac{\lambda_1 T_1}{Qa} \simeq \frac{1}{\pi^{1/2}}\left(\beta^{-1} - \frac{1}{2}\beta\right)\tau + (1-\beta^{-2}-\chi)$$
$$- e^{\beta^2\tau^2/4}\,\mathrm{erfc}\left(\frac{\beta\tau}{2}\right)\left[1-\beta^{-2}-\chi-\frac{1}{2}(1-\chi)^2 - \frac{\beta^2\tau^2}{4}\right]. \tag{14}$$

Note that, in order to obtain approximations which are uniformly valid for small β, the denominators in equation (13) are not further expanded. In a similar fashion, we obtain from equation (5), the approximation

$$T_2 = \frac{Q}{p^{3/2}k_1\lambda_1} \frac{e^{-p^{1/2}k_2(x-a)}}{\beta + p^{1/2}k_1a}\left[1 - \frac{\beta p k_1^2 a^2}{\beta + p^{1/2}k_1 a}\right] \tag{15}$$

which gives

$$\Theta_2 = \frac{\lambda_1 T_2}{Qa} \simeq \frac{1}{\pi^{1/2}}\left(\beta^{-1} - \frac{1}{2}\beta\right)\tau \exp\left(-\frac{\xi^2}{\tau^2}\right)$$
$$-(\beta^{-2}+\xi\beta^{-1})\,\mathrm{erfc}\left(\frac{\xi}{\tau}\right) + \exp\left(\beta\xi + \beta^2\frac{\tau^2}{4}\right)$$
$$\times\,\mathrm{erfc}\left(\frac{\xi}{\tau} + \frac{\beta\tau}{2}\right)\left(\beta^{-2} + \frac{1}{2}\beta^2\tau^2 + \xi\right). \tag{16}$$

We note the following special cases. For $\beta = 0$ and $\tau \gg 1$

$$\Theta_1 \simeq \tfrac{1}{4}\tau^2 + (1-\chi)^2 - \tfrac{1}{6} \tag{17}$$

$$\Theta_2 \simeq \left(\frac{1}{4}\tau^2 + \frac{1}{2}\xi^2 - \frac{1}{6}\right)\mathrm{erfc}\left(\frac{\xi}{\tau}\right) - \frac{\xi\tau}{\pi^{1/2}}\exp\left(-\frac{\xi^2}{\tau^2}\right). \tag{18}$$

For $\beta \to \infty$ and $\tau \gg 1$

$$\Theta_1 \simeq 1-\chi, \quad \Theta_2 = 0. \tag{19}$$

Also, although equations (14) and (16) are uniformly valid approximations for all β, they can be further simplified when $\beta\tau \gg 1$. The results are

$$\Theta_1 \simeq \frac{\tau}{\pi^{1/2}\beta} + 1 - \chi - \beta^{-2} - \left(1 - \frac{1}{2}\chi^2 - \beta^{-2}\right)\frac{2}{\pi^{1/2}\beta\tau} \tag{20}$$

$$\Theta_2 \simeq \frac{\tau}{\pi^{1/2}\beta}\exp\left(-\frac{\xi^2}{\tau^2}\right) - \beta^{-1}(\xi + \beta^{-1})\,\mathrm{erfc}\left(\frac{\xi}{\tau}\right) - \frac{(1-2\beta^{-2})}{\pi^{1/2}\beta\tau}\exp\left(-\frac{\xi^2}{\tau^2}\right). \tag{21}$$

2.2. *Interpretation by contour integral*

Expressions (4) and (5) can also be interpreted in the usual way by using the contour integral form of the inverse transform. In order to avoid complications near the origin in the complex p-plane, it is necessary to subtract the most

† The inverse transforms required are all listed in the more comprehensive tables, see for example Abramowitz and Stegun [6].

significant part of the singularity for small $|p|$ (and hence for large τ). Otherwise, the interpretation proceeds in the conventional fashion by a deformation of the contour along the two sides of the negative real axis. The details are omitted; the final results are as follows

$$\Theta_1 = \frac{\tau}{\pi^{1/2}\beta} + 1 - \chi - \beta^{-2} - \frac{2}{\pi\beta} \times \int_0^\infty \left(\frac{\cos(\chi s)}{1-(1-\beta^{-2})\sin^2 s} - 1 \right) \exp\left(-\frac{\tau^2 s^2}{4}\right) \frac{ds}{s^2} \quad (22)$$

$$\Theta_2 = \frac{\tau}{\pi^{1/2}\beta} \exp\left(-\frac{\xi^2}{\tau^2}\right) - \beta^{-1}(\xi+\beta^{-1}) \operatorname{erfc}\left(\frac{\xi}{\tau}\right) - \frac{2}{\pi\beta} \int_0^\infty \left[\frac{\cos s \cos \xi s - \beta^{-1} \sin s \sin \xi s}{1-(1-\beta^{-2})\sin^2 s} - \cos \xi s + \beta^{-1} s \sin \xi s \right] \exp\left(-\frac{\tau^2 s^2}{4}\right) \frac{ds}{s^2}. \quad (23)$$

Relation (23) can in fact be further simplified to

$$\Theta_2 = \beta^{-1}\left(\frac{\tau}{\pi^{1/2}} - \xi\right) - \beta^{-2} - \frac{2}{\pi\beta} \int_0^\infty \left[\frac{\cos s \cos \xi s - \beta^{-1} \sin s \sin \xi s}{1-(1-\beta^{-2})\sin^2 s} - 1 \right] \times \exp\left(-\frac{\tau^2 s^2}{4}\right) \frac{ds}{s^2} \quad (24)$$

but the integrated part of the expression is then not a uniformly valid approximation of $\xi \gg 1, \tau \gg 1$.

It should be noted that two non-dimensional space coordinates are used, namely $\chi = x/a$ for $0 < x < a$ and $\xi = k_2/k_1(x/a - 1)$ for $a < x < \infty$. This should be kept in mind in the assessment of Figs. 3(a)–(c) below. The reason for the change of coordinates is that information can be presented in a very general form, without the necessity of choosing particular values for the parameters. In fact, the original problem, as presented, involves two independent variables, x and t, and eight parameters, $\rho_1, \rho_2, c_1, c_2, \lambda_1, \lambda_2, Q$ and a. By contrast the final formulae for the non-dimensional temperatures θ_1 and θ_2 involve two non-dimensional variables χ (or ξ) and τ and only one parameter β.

3. NUMERICAL CALCULATIONS

Preliminary calculations were carried out using the integral expressions (22) and (23). They were made on a PDP 11 with the help of the I.M.S. library. The results of these computations were used to check the range of validity of the various alternative formulae. The calculation of these formulae which is much quicker than that of the integral expressions, has been made on a microcomputer Apple II.

If we require the error to be inferior to 1%, the ranges of validity for the various formulae are as follows.

Equations (9) and (11)

These equations are valid for any value of β and τ. However, the computing time is substantially longer for them than it is with the approximate solutions. For a precision of 1%, the use of equations (9) and (11) is required only

for $\tau \leqslant 2$ if $\beta = 0$

for $\tau \leqslant 7$ if $0 < \beta < 1$

for $\tau \leqslant 4$ if $1 \leqslant \beta < \infty$

for $\tau \leqslant 3$ if $\beta = \infty$.

When equations (9) and (11) are used in the ranges indicated above, it is observed that it is sufficient to consider $(\tau + 1)$ terms in the summation.

Equations (14) and (16)

for $0 < \beta < 1$ and $7 \leqslant \tau \leqslant \infty$.

Equations (17) and (18)

for $\beta = 0$ and $2 \leqslant \tau \leqslant \infty$.

Equation (19)

for $\beta = \infty$ $3 \leqslant \tau \leqslant \infty$.

Equations (20) and (21)

for $1 \leqslant \beta \leqslant \infty$ and $4 \leqslant \tau \leqslant \infty$.

On Fig. 1, we have plotted the variation of the interface temperature as a function of time with β as a parameter. Small values of β essentially correspond to the case of an insulator mechanically protected by a conducting layer, whereas large values of β correspond to the case of a conductor thermally protected by an insulating layer. An interesting feature of the solution is the difference in temperature between the surface and in the interface. This difference is illustrated on Fig. 2. From equation (20) we have $(\theta_1)_{\chi=0} - (\theta_1)_{\chi=1} \to 1$ as $\beta\tau \to \infty$ and this can be verified on the figure. Numerous calculations have been made on the temperature distribution in the two-layer medium as a function of time. As an example, Fig. 3 shows the results obtained for the particular values of the effusivity ratio β (0, 1 and ∞). Note that $\beta = 1$ does not necessarily correspond to the case of a homogeneous medium, and that the solution for $\beta = 0$ is not trivial. For $\beta_1 = \infty$, $\theta_2 = 0$ in the semi-infinite medium and we observe that θ_1 tends very quickly to the asymptotic value $(1 - \chi)$.

4. CONCLUSIONS

Alternative exact solutions for the transient heat conduction through a two-layer medium have been given in dimensionless form and for the complete range of the effusivity

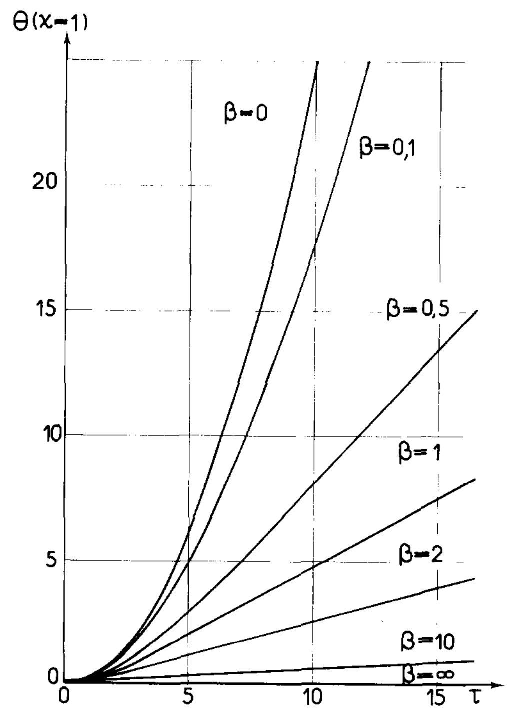


FIG. 1. Interface temperature as a function of time (β is the effusivity ratio of the two media).

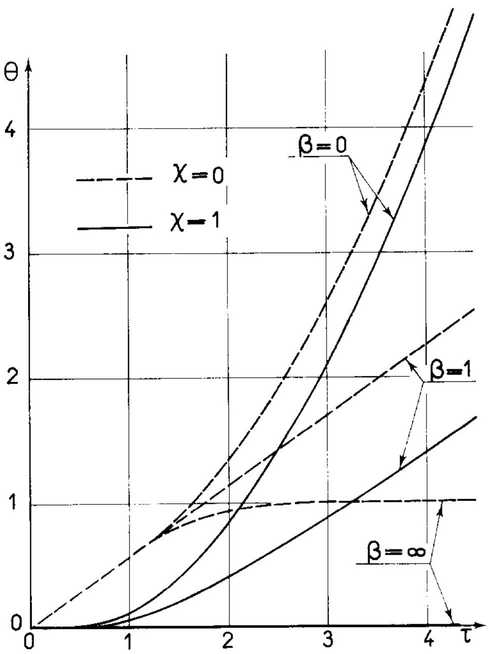


FIG. 2. Temperature of the surface on which the constant flux is applied ($\chi = 0$) and interface ($\chi = 1$) temperature.

ratio of the two media. Solutions in the form of rapidly converging series have been derived. Their range of applicability has been determined by comparison with the exact solution. When confronted with a particular problem, this result enables one to immediately judge which approximate formula is appropriate. In some practical cases, the solution can be calculated with good accuracy on a pocket calculator.

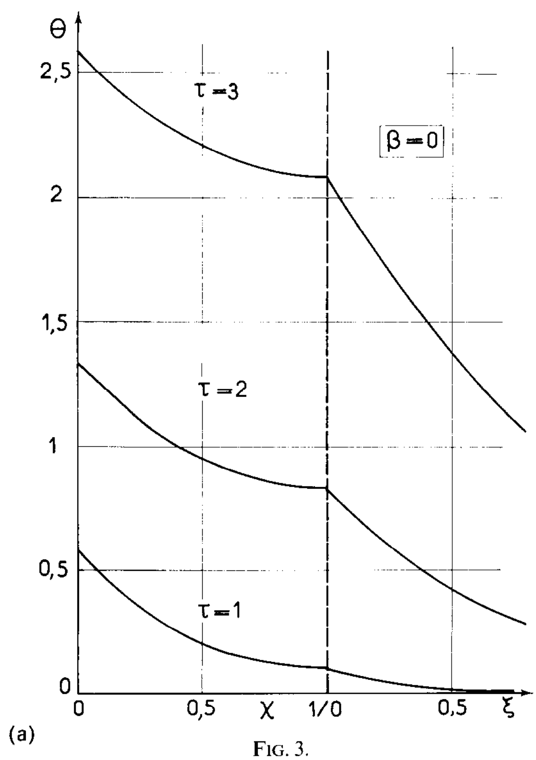
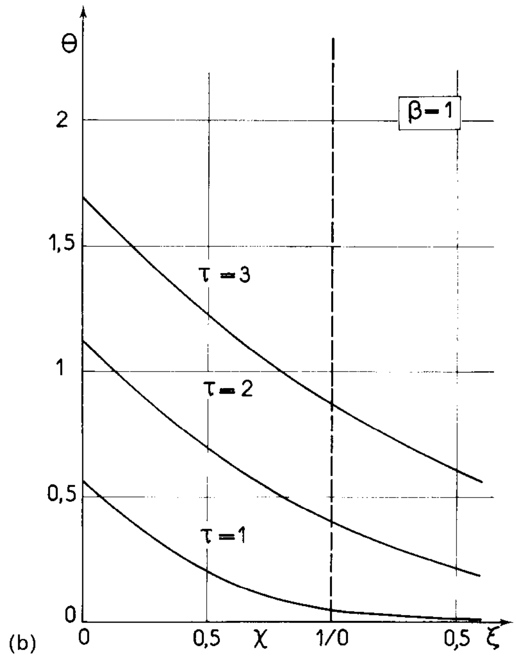


FIG. 3.



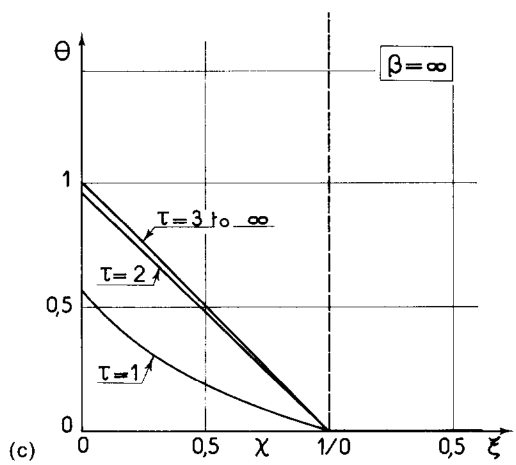


FIG. 3. Temperature distribution within the two media as a function of time: (a) $\beta = 0$; (b) $\beta = 1$; (c) $\beta = \infty$.

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